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Some Properties of Restrictions of Finite Sequences

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Summary. The aim of the paper is to define some basic notions of restrictions of finite sequences.

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The articles [6], [8], [1], [9], [3], [2], [7], [5], and [4] provide the notation and terminology for this paper.

In this paper i, j, k, n are natural numbers.

One can prove the following propositions:

(1) If $i \leq n$, then $(n-i)+1$ is a natural number.

(2) If $i \in \mathrm{Seg}\, n$, then $(n-i)+1 \in \mathrm{Seg}\, n$.

(3) For every function f and for all sets x, y such that $f^{-1}(\{y\}) = \{x\}$ holds $x \in \mathrm{dom}\, f$ and $y \in \mathrm{rng}\, f$ and $f(x) = y$.

(4) For every function f holds f is one-to-one iff for every set x such that $x \in \mathrm{dom}\, f$ holds $f^{-1}(\{f(x)\}) = \{x\}$.

(5) For every function f and for all sets y_1, y_2 such that f is one-to-one and $y_1 \in \mathrm{rng}\, f$ and $y_2 \in \mathrm{rng}\, f$ and $f^{-1}(\{y_1\}) = f^{-1}(\{y_2\})$ holds $y_1 = y_2$.

Let x be a set. Observe that $\langle x \rangle$ is non empty.

Let us note that every set which is empty is also trivial.

Let x be a set. One can check that $\langle x \rangle$ is trivial. Let y be a set. Note that $\langle x, y \rangle$ is non trivial.

Let us note that there exists a finite sequence which is one-to-one and non empty.

The following propositions are true:

(6) For every non empty finite sequence f holds $1 \in \mathrm{dom}\, f$ and $\mathrm{len}\, f \in \mathrm{dom}\, f$.

(7) For every non empty finite sequence f there exists i such that $i+1 = \mathrm{len}\, f$.

(8) For every set x and for every finite sequence f holds $\mathrm{len}(\langle x \rangle ^\frown f) = 1 + \mathrm{len}\, f$.

The scheme *domSeqLambda* deals with a natural number $\mathcal{A}$ and a unary functor $\mathcal{F}$ yielding a set, and states that:

> There exists a finite sequence p such that $\mathrm{len}\, p = \mathcal{A}$ and for every k such that $k \in \mathrm{dom}\, p$ holds $p(k) = \mathcal{F}(k)$

for all values of the parameters.

We now state three propositions:

(10)[1] For every finite sequence f and for all sets p, q such that $p \in \mathrm{rng}\, f$ and $q \in \mathrm{rng}\, f$ and

[1] The proposition (9) has been removed.

$p \leftrightarrow f = q \leftrightarrow f$ holds $p = q$.

(11) For all finite sequences f, g such that $n+1 \in \operatorname{dom} f$ and $g = f \restriction \operatorname{Seg} n$ holds $f \restriction \operatorname{Seg}(n+1) = g \frown \langle f(n+1) \rangle$.

(12) For every one-to-one finite sequence f such that $i \in \operatorname{dom} f$ holds $f(i) \leftrightarrow f = i$.

In the sequel D denotes a non empty set, p denotes an element of D, and f, g denote finite sequences of elements of D.

Let D be a non empty set. One can check that there exists a finite sequence of elements of D which is one-to-one and non empty.

One can prove the following propositions:

(13) If $\operatorname{dom} f = \operatorname{dom} g$ and for every i such that $i \in \operatorname{dom} f$ holds $f_i = g_i$, then $f = g$.

(14) If $\operatorname{len} f = \operatorname{len} g$ and for every k such that $1 \leq k$ and $k \leq \operatorname{len} f$ holds $f_k = g_k$, then $f = g$.

(15) If $\operatorname{len} f = 1$, then $f = \langle f_1 \rangle$.

(16) Let D be a non empty set, p be an element of D, and f be a finite sequence of elements of D. Then $(\langle p \rangle \frown f)_1 = p$.

(18)[2] For every set D and for every finite sequence f of elements of D holds $\operatorname{len}(f \restriction i) \leq \operatorname{len} f$.

(19) For every set D and for every finite sequence f of elements of D holds $\operatorname{len}(f \restriction i) \leq i$.

(20) For every set D and for every finite sequence f of elements of D holds $\operatorname{dom}(f \restriction i) \subseteq \operatorname{dom} f$.

(21) $\operatorname{rng}(f \restriction i) \subseteq \operatorname{rng} f$.

(23)[3] For every set D and for every finite sequence f of elements of D such that f is non empty holds $f \restriction 1 = \langle f_1 \rangle$.

(24) If $i+1 = \operatorname{len} f$, then $f = (f \restriction i) \frown \langle f_{\operatorname{len} f} \rangle$.

Let us consider i, D and let f be an one-to-one finite sequence of elements of D. One can verify that $f \restriction i$ is one-to-one.

We now state a number of propositions:

(25) For every set D and for all finite sequences f, g of elements of D such that $i \leq \operatorname{len} f$ holds $(f \frown g) \restriction i = f \restriction i$.

(26) For every set D and for all finite sequences f, g of elements of D holds $(f \frown g) \restriction \operatorname{len} f = f$.

(27) For every set D and for every finite sequence f of elements of D such that $p \in \operatorname{rng} f$ holds $(f \leftarrow p) \frown \langle p \rangle = f \restriction p \leftrightarrow f$.

(28) $\operatorname{len}(f_{\downharpoonright i}) \leq \operatorname{len} f$.

(29) If $i \in \operatorname{dom}(f_{\downharpoonright n})$, then $n+i \in \operatorname{dom} f$.

(30) If $i \in \operatorname{dom}(f_{\downharpoonright n})$, then $(f_{\downharpoonright n})_i = f_{n+i}$.

(31) $f_{\downharpoonright 0} = f$.

(32) If f is non empty, then $f = \langle f_1 \rangle \frown (f_{\downharpoonright 1})$.

(33) If $i+1 = \operatorname{len} f$, then $f_{\downharpoonright i} = \langle f_{\operatorname{len} f} \rangle$.

(34) If $j+1 = i$ and $i \in \operatorname{dom} f$, then $\langle f_i \rangle \frown (f_{\downharpoonright i}) = f_{\downharpoonright j}$.

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[2] The proposition (17) has been removed.

[3] The proposition (22) has been removed.

(35) For every set D and for every finite sequence f of elements of D such that $\operatorname{len} f \leq i$ holds $f_{\downharpoonright i}$ is empty.

(36) $\operatorname{rng}(f_{\downharpoonright n}) \subseteq \operatorname{rng} f$.

Let us consider i, D and let f be an one-to-one finite sequence of elements of D. Observe that $f_{\downharpoonright i}$ is one-to-one.

Next we state several propositions:

(37) If f is one-to-one, then $\operatorname{rng}(f \upharpoonright n)$ misses $\operatorname{rng}(f_{\downharpoonright n})$.

(38) If $p \in \operatorname{rng} f$, then $f \rightarrow p = f_{\downharpoonright p \hookleftarrow f}$.

(39) $(f \frown g)_{\downharpoonright \operatorname{len} f + i} = g_{\downharpoonright i}$.

(40) $(f \frown g)_{\downharpoonright \operatorname{len} f} = g$.

(41) If $p \in \operatorname{rng} f$, then $f_{p \hookleftarrow f} = p$.

(42) If $i \in \operatorname{dom} f$, then $f_i \hookleftarrow f \leq i$.

(43) If $p \in \operatorname{rng}(f \upharpoonright i)$, then $p \hookleftarrow (f \upharpoonright i) = p \hookleftarrow f$.

(44) If $i \in \operatorname{dom} f$ and f is one-to-one, then $f_i \hookleftarrow f = i$.

Let us consider D, f and let p be a set. The functor $f -: p$ yields a finite sequence of elements of D and is defined by:

(Def. 1) $f -: p = f \upharpoonright p \hookleftarrow f$.

Next we state several propositions:

(45) If $p \in \operatorname{rng} f$, then $\operatorname{len}(f -: p) = p \hookleftarrow f$.

(46) If $p \in \operatorname{rng} f$ and $i \in \operatorname{Seg}(p \hookleftarrow f)$, then $(f -: p)_i = f_i$.

(47) If $p \in \operatorname{rng} f$, then $(f -: p)_1 = f_1$.

(48) If $p \in \operatorname{rng} f$, then $(f -: p)_{p \hookleftarrow f} = p$.

(49) For every set x such that $x \in \operatorname{rng} f$ and $p \in \operatorname{rng} f$ and $x \hookleftarrow f \leq p \hookleftarrow f$ holds $x \in \operatorname{rng}(f -: p)$.

(50) If $p \in \operatorname{rng} f$, then $f -: p$ is non empty.

(51) $\operatorname{rng}(f -: p) \subseteq \operatorname{rng} f$.

Let us consider D, p and let f be an one-to-one finite sequence of elements of D. Note that $f -: p$ is one-to-one.

Let us consider D, f, p. The functor $f :- p$ yielding a finite sequence of elements of D is defined as follows:

(Def. 2) $f :- p = \langle p \rangle \frown (f_{\downharpoonright p \hookleftarrow f})$.

Next we state three propositions:

(52) If $p \in \operatorname{rng} f$, then there exists i such that $i + 1 = p \hookleftarrow f$ and $f :- p = f_{\downharpoonright i}$.

(53) If $p \in \operatorname{rng} f$, then $\operatorname{len}(f :- p) = (\operatorname{len} f - p \hookleftarrow f) + 1$.

(54) If $p \in \operatorname{rng} f$ and $j + 1 \in \operatorname{dom}(f :- p)$, then $j + p \hookleftarrow f \in \operatorname{dom} f$.

Let us consider D, p, f. Note that $f :- p$ is non empty.

Next we state several propositions:

(55) If $p \in \operatorname{rng} f$ and $j + 1 \in \operatorname{dom}(f :- p)$, then $(f :- p)_{j+1} = f_{j + p \hookleftarrow f}$.

(56) $(f :- p)_1 = p$.

(57) If $p \in \operatorname{rng} f$, then $(f :- p)_{\operatorname{len}(f :- p)} = f_{\operatorname{len} f}$.

(58) If $p \in \operatorname{rng} f$, then $\operatorname{rng}(f :- p) \subseteq \operatorname{rng} f$.

(59) If $p \in \operatorname{rng} f$ and f is one-to-one, then $f :- p$ is one-to-one.

Let f be a finite sequence. The functor $\operatorname{Rev}(f)$ yielding a finite sequence is defined as follows:

(Def. 3) $\operatorname{len}\operatorname{Rev}(f) = \operatorname{len} f$ and for every i such that $i \in \operatorname{dom}\operatorname{Rev}(f)$ holds $(\operatorname{Rev}(f))(i) = f((\operatorname{len} f - i) + 1)$.

The following three propositions are true:

(60) For every finite sequence f holds $\operatorname{dom} f = \operatorname{dom}\operatorname{Rev}(f)$ and $\operatorname{rng} f = \operatorname{rng}\operatorname{Rev}(f)$.

(61) For every finite sequence f such that $i \in \operatorname{dom} f$ holds $(\operatorname{Rev}(f))(i) = f((\operatorname{len} f - i) + 1)$.

(62) For every finite sequence f and for all natural numbers i, j such that $i \in \operatorname{dom} f$ and $i + j = \operatorname{len} f + 1$ holds $j \in \operatorname{dom}\operatorname{Rev}(f)$.

Let f be an empty finite sequence. One can check that $\operatorname{Rev}(f)$ is empty.
Next we state three propositions:

(63) For every set x holds $\operatorname{Rev}(\langle x \rangle) = \langle x \rangle$.

(64) For all sets x_1, x_2 holds $\operatorname{Rev}(\langle x_1, x_2 \rangle) = \langle x_2, x_1 \rangle$.

(65) For every finite sequence f holds $f(1) = (\operatorname{Rev}(f))(\operatorname{len} f)$ and $f(\operatorname{len} f) = (\operatorname{Rev}(f))(1)$.

Let f be an one-to-one finite sequence. One can check that $\operatorname{Rev}(f)$ is one-to-one.
One can prove the following propositions:

(66) For every finite sequence f and for every set x holds $\operatorname{Rev}(f \frown \langle x \rangle) = \langle x \rangle \frown \operatorname{Rev}(f)$.

(67) For all finite sequences f, g holds $\operatorname{Rev}(f \frown g) = (\operatorname{Rev}(g)) \frown \operatorname{Rev}(f)$.

Let us consider D, f. Then $\operatorname{Rev}(f)$ is a finite sequence of elements of D.
One can prove the following propositions:

(68) If f is non empty, then $f_1 = (\operatorname{Rev}(f))_{\operatorname{len} f}$ and $f_{\operatorname{len} f} = (\operatorname{Rev}(f))_1$.

(69) If $i \in \operatorname{dom} f$ and $i + j = \operatorname{len} f + 1$, then $f_i = (\operatorname{Rev}(f))_j$.

Let us consider D, f, p, n. The functor $\operatorname{Ins}(f, n, p)$ yielding a finite sequence of elements of D is defined by:

(Def. 4) $\operatorname{Ins}(f, n, p) = (f \restriction n) \frown \langle p \rangle \frown (f_{\downharpoonright n})$.

We now state four propositions:

(70) $\operatorname{Ins}(f, 0, p) = \langle p \rangle \frown f$.

(71) If $\operatorname{len} f \leq n$, then $\operatorname{Ins}(f, n, p) = f \frown \langle p \rangle$.

(72) $\operatorname{len}\operatorname{Ins}(f, n, p) = \operatorname{len} f + 1$.

(73) $\operatorname{rng}\operatorname{Ins}(f, n, p) = \{p\} \cup \operatorname{rng} f$.

Let us consider D, f, n, p. Note that $\operatorname{Ins}(f, n, p)$ is non empty.
Next we state several propositions:

(74) $p \in \operatorname{rng}\operatorname{Ins}(f, n, p)$.

(75) If $i \in \operatorname{dom}(f \restriction n)$, then $(\operatorname{Ins}(f,n,p))_i = f_i$.

(76) If $n \leq \operatorname{len} f$, then $(\operatorname{Ins}(f,n,p))_{n+1} = p$.

(77) If $n+1 \leq i$ and $i \leq \operatorname{len} f$, then $(\operatorname{Ins}(f,n,p))_{i+1} = f_i$.

(78) If $1 \leq n$ and f is non empty, then $(\operatorname{Ins}(f,n,p))_1 = f_1$.

(79) If f is one-to-one and $p \notin \operatorname{rng} f$, then $\operatorname{Ins}(f,n,p)$ is one-to-one.

REFERENCES

[1] Grzegorz Bancerek. The fundamental properties of natural numbers. *Journal of Formalized Mathematics*, 1, 1989. http://mizar.org/JFM/Vol1/nat_1.html.

[2] Grzegorz Bancerek and Krzysztof Hryniewiecki. Segments of natural numbers and finite sequences. *Journal of Formalized Mathematics*, 1, 1989. http://mizar.org/JFM/Vol1/finseq_1.html.

[3] Czesław Byliński. Functions and their basic properties. *Journal of Formalized Mathematics*, 1, 1989. http://mizar.org/JFM/Vol1/funct_1.html.

[4] Agata Darmochwał and Yatsuka Nakamura. The topological space $\mathcal{E}^2_T$. Arcs, line segments and special polygonal arcs. *Journal of Formalized Mathematics*, 3, 1991. http://mizar.org/JFM/Vol3/topreal1.html.

[5] Jarosław Kotowicz. Functions and finite sequences of real numbers. *Journal of Formalized Mathematics*, 5, 1993. http://mizar.org/JFM/Vol5/rfinseq.html.

[6] Andrzej Trybulec. Tarski Grothendieck set theory. *Journal of Formalized Mathematics*, Axiomatics, 1989. http://mizar.org/JFM/Axiomatics/tarski.html.

[7] Wojciech A. Trybulec. Pigeon hole principle. *Journal of Formalized Mathematics*, 2, 1990. http://mizar.org/JFM/Vol2/finseq_4.html.

[8] Zinaida Trybulec. Properties of subsets. *Journal of Formalized Mathematics*, 1, 1989. http://mizar.org/JFM/Vol1/subset_1.html.

[9] Edmund Woronowicz. Relations and their basic properties. *Journal of Formalized Mathematics*, 1, 1989. http://mizar.org/JFM/Vol1/relat_1.html.

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