Basic Properties of Even and Odd Functions

Bo Li
Qingdao University of Science and Technology
China

Yanhong Men
Qingdao University of Science and Technology
China

Summary. In this article we present definitions, basic properties and some examples of even and odd functions [6].

MML identifier: FUNCT_8, version: 7.11.02 4.120.1050

The articles [2], [5], [1], [8], [14], [12], [15], [7], [17], [3], [4], [11], [19], [13], [10], [18], [16], and [9] provide the notation and terminology for this paper.

1. Even and Odd Functions

In this paper \( x, r \) are real numbers.
Let \( A \) be a set. We say that \( A \) is symmetrical if and only if:

(Def. 1) For every complex number \( x \) such that \( x \in A \) holds \(-x \in A\).
Let us note that there exists a subset of \( \mathbb{C} \) which is symmetrical.
Let us observe that there exists a subset of \( \mathbb{R} \) which is symmetrical.
In the sequel \( A \) denotes a symmetrical subset of \( \mathbb{C} \).
Let \( R \) be a binary relation. We say that \( R \) has symmetrical domain if and only if:

(Def. 2) \( \text{dom} \ R \) is symmetrical.

Let us observe that every binary relation which is empty has also symmetrical domain and there exists a binary relation which has symmetrical domain.
Let \( R \) be a binary relation with symmetrical domain. Observe that \( \text{dom} \ R \) is symmetrical.

Let \( X, Y \) be complex-membered sets and let \( F \) be a partial function from \( X \) to \( Y \). We say that \( F \) is quasi even if and only if:
(Def. 3) For every $x$ such that $x, -x \in \text{dom } F$ holds $F(-x) = F(x)$.

Let $X, Y$ be complex-membered sets and let $F$ be a partial function from $X$ to $Y$. We say that $F$ is even if and only if:

(Def. 4) $F$ is quasi even and has symmetrical domain.

Let $X, Y$ be complex-membered sets. Note that every partial function from $X$ to $Y$ which is quasi even and has symmetrical domain is also even and every partial function from $X$ to $Y$ which is even is also quasi even and has symmetrical domain.

Let $A$ be a set, let $X, Y$ be complex-membered sets, and let $F$ be a partial function from $X$ to $Y$. We say that $F$ is even on $A$ if and only if:

(Def. 5) $A \subseteq \text{dom } F$ and $F|A$ is even.

Let $X, Y$ be complex-membered sets and let $F$ be a partial function from $X$ to $Y$. We say that $F$ is quasi odd if and only if:

(Def. 6) For every $x$ such that $x, -x \in \text{dom } F$ holds $F(-x) = -F(x)$.

Let $X, Y$ be complex-membered sets and let $F$ be a partial function from $X$ to $Y$. We say that $F$ is odd if and only if:

(Def. 7) $F$ is quasi odd and has symmetrical domain.

Let $X, Y$ be complex-membered sets. Note that every partial function from $X$ to $Y$ which is quasi odd and has symmetrical domain is also odd and every partial function from $X$ to $Y$ which is odd is also quasi odd and has symmetrical domain.

Let $A$ be a set, let $X, Y$ be complex-membered sets, and let $F$ be a partial function from $X$ to $Y$. We say that $F$ is odd on $A$ if and only if:

(Def. 8) $A \subseteq \text{dom } F$ and $F|A$ is odd.

In the sequel $F, G$ denote partial functions from $\mathbb{R}$ to $\mathbb{R}$.

The following propositions are true:

(1) $F$ is odd on $A$ if and only if $A \subseteq \text{dom } F$ and for every $x$ such that $x \in A$ holds $F(x) + F(-x) = 0$.

(2) $F$ is even on $A$ if and only if $A \subseteq \text{dom } F$ and for every $x$ such that $x \in A$ holds $F(x) - F(-x) = 0$.

(3) If $F$ is odd on $A$ and for every $x$ such that $x \in A$ holds $F(x) \neq 0$, then $A \subseteq \text{dom } F$ and for every $x$ such that $x \in A$ holds $\frac{F(x)}{F(-x)} = -1$.

(4) If $A \subseteq \text{dom } F$ and for every $x$ such that $x \in A$ holds $F(x) \neq 0$, then $F$ is odd on $A$.

(5) If $F$ is even on $A$ and for every $x$ such that $x \in A$ holds $F(x) \neq 0$, then $A \subseteq \text{dom } F$ and for every $x$ such that $x \in A$ holds $\frac{F(x)}{F(-x)} = 1$.

(6) If $A \subseteq \text{dom } F$ and for every $x$ such that $x \in A$ holds $\frac{F(x)}{F(-x)} = 1$, then $F$ is even on $A$. 
(7) If $F$ is even on $A$ and odd on $A$, then for every $x$ such that $x \in A$ holds $F(x) = 0$.

(8) If $F$ is even on $A$, then for every $x$ such that $x \in A$ holds $F(x) = F(|x|)$.

(9) If $A \subseteq \text{dom } F$ and for every $x$ such that $x \in A$ holds $F(x) = F(|x|)$, then $F$ is even on $A$.

(10) If $F$ is odd on $A$ and $G$ is odd on $A$, then $F + G$ is odd on $A$.

(11) If $F$ is even on $A$ and $G$ is even on $A$, then $F + G$ is even on $A$.

(12) If $F$ is odd on $A$ and $G$ is odd on $A$, then $F - G$ is odd on $A$.

(13) If $F$ is even on $A$ and $G$ is even on $A$, then $F - G$ is odd on $A$.

(14) If $F$ is odd on $A$, then $rF$ is odd on $A$.

(15) If $F$ is even on $A$, then $rF$ is even on $A$.

(16) If $F$ is odd on $A$, then $-F$ is odd on $A$.

(17) If $F$ is even on $A$, then $-F$ is even on $A$.

(18) If $F$ is odd on $A$, then $F^{-1}$ is odd on $A$.

(19) If $F$ is even on $A$, then $F^{-1}$ is even on $A$.

(20) If $F$ is odd on $A$, then $|F|$ is even on $A$.

(21) If $F$ is even on $A$, then $|F|$ is even on $A$.

(22) If $F$ is odd on $A$ and $G$ is odd on $A$, then $FG$ is even on $A$.

(23) If $F$ is even on $A$ and $G$ is even on $A$, then $FG$ is even on $A$.

(24) If $F$ is even on $A$ and $G$ is odd on $A$, then $FG$ is odd on $A$.

(25) If $F$ is even on $A$, then $r + F$ is even on $A$.

(26) If $F$ is even on $A$, then $F - r$ is even on $A$.

(27) If $F$ is even on $A$, then $F^2$ is even on $A$.

(28) If $F$ is odd on $A$, then $F^2$ is even on $A$.

(29) If $F$ is odd on $A$ and $G$ is odd on $A$, then $F/G$ is even on $A$.

(30) If $F$ is even on $A$ and $G$ is even on $A$, then $F/G$ is even on $A$.

(31) If $F$ is odd on $A$ and $G$ is even on $A$, then $F/G$ is odd on $A$.

(32) If $F$ is even on $A$ and $G$ is odd on $A$, then $F/G$ is odd on $A$.

(33) If $F$ is odd, then $-F$ is odd.

(34) If $F$ is even, then $-F$ is even.

(35) If $F$ is odd, then $F^{-1}$ is odd.

(36) If $F$ is even, then $F^{-1}$ is even.

(37) If $F$ is odd, then $|F|$ is even.

(38) If $F$ is even, then $|F|$ is even.

(39) If $F$ is odd, then $F^2$ is even.

(40) If $F$ is even, then $F^2$ is even.

(41) If $F$ is even, then $r + F$ is even.
(42) If $F$ is even, then $F - r$ is even.
(43) If $F$ is odd, then $r F$ is odd.
(44) If $F$ is even, then $r F$ is even.
(45) If $F$ is odd and $G$ is odd and $\text{dom } F \cap \text{dom } G$ is symmetrical, then $F + G$ is odd.
(46) If $F$ is even and $G$ is even and $\text{dom } F \cap \text{dom } G$ is symmetrical, then $F + G$ is even.
(47) If $F$ is odd and $G$ is odd and $\text{dom } F \cap \text{dom } G$ is symmetrical, then $F - G$ is odd.
(48) If $F$ is even and $G$ is even and $\text{dom } F \cap \text{dom } G$ is symmetrical, then $F - G$ is even.
(49) If $F$ is odd and $G$ is odd and $\text{dom } F \cap \text{dom } G$ is symmetrical, then $FG$ is even.
(50) If $F$ is even and $G$ is even and $\text{dom } F \cap \text{dom } G$ is symmetrical, then $FG$ is even.
(51) If $F$ is even and $G$ is odd and $\text{dom } F \cap \text{dom } G$ is symmetrical, then $F/G$ is odd.
(52) If $F$ is odd and $G$ is odd and $\text{dom } F \cap \text{dom } G$ is symmetrical, then $F/G$ is even.
(53) If $F$ is even and $G$ is even and $\text{dom } F \cap \text{dom } G$ is symmetrical, then $F/G$ is even.
(54) If $F$ is odd and $G$ is even and $\text{dom } F \cap \text{dom } G$ is symmetrical, then $F/G$ is odd.
(55) If $F$ is even and $G$ is odd and $\text{dom } F \cap \text{dom } G$ is symmetrical, then $F/G$ is odd.

2. Some Examples

The function signum from $\mathbb{R}$ into $\mathbb{R}$ is defined as follows:

(Def. 9) For every real number $x$ holds $(\text{signum})(x) = sgn x$.

Let $x$ be a real number. Observe that $(\text{signum})(x)$ is real.

Next we state a number of propositions:

(56) For every real number $x$ such that $x > 0$ holds $(\text{signum})(x) = 1$.
(57) For every real number $x$ such that $x < 0$ holds $(\text{signum})(x) = -1$.
(58) $(\text{signum})(0) = 0$.
(59) For every real number $x$ holds $(\text{signum})(-x) = -(\text{signum})(x)$.
(60) For every symmetrical subset $A$ of $\mathbb{R}$ holds signum is odd on $A$.
(61) For every real number $x$ such that $x \geq 0$ holds $\|\|=\mathbb{R}(x) = x$. 
(62) For every real number \( x \) such that \( x < 0 \) holds \( |x|_R(x) = -x \).

(63) For every real number \( x \) holds \( |x|_R(-x) = |x|_R(x) \).

(64) For every symmetrical subset \( A \) of \( \mathbb{R} \) holds \( |x|_R \) is even on \( A \).

(65) For every symmetrical subset \( A \) of \( \mathbb{R} \) holds the function \( \sin \) is odd on \( A \).

(66) For every symmetrical subset \( A \) of \( \mathbb{R} \) holds the function \( \cos \) is even on \( A \).

One can verify that the function \( \sin \) is odd.

Let us observe that the function \( \cos \) is even.

The following two propositions are true:

(67) For every symmetrical subset \( A \) of \( \mathbb{R} \) holds the function \( \sinh \) is odd on \( A \).

(68) For every symmetrical subset \( A \) of \( \mathbb{R} \) holds the function \( \cosh \) is even on \( A \).

Let us mention that the function \( \sinh \) is odd.

Let us mention that the function \( \cosh \) is even.

Next we state a number of propositions:

(69) If \( A \subseteq [-\pi/2, \pi/2] \), then the function \( \tan \) is odd on \( A \).

(70) Suppose \( A \subseteq \text{dom (the function tan)} \) and for every \( x \) such that \( x \in A \) holds \( \text{(the function cos)}(x) \neq 0 \). Then the function \( \tan \) is odd on \( A \).

(71) Suppose \( A \subseteq \text{dom (the function cot)} \) and for every \( x \) such that \( x \in A \) holds \( \text{(the function sin)}(x) \neq 0 \). Then the function \( \cot \) is odd on \( A \).

(72) If \( A \subseteq [-1, 1] \), then the function \( \text{arctan} \) is odd on \( A \).

(73) For every symmetrical subset \( A \) of \( \mathbb{R} \) holds \(|\text{the function sin} \mid \) is even on \( A \).

(74) For every symmetrical subset \( A \) of \( \mathbb{R} \) holds \(|\text{the function cos} \mid \) is even on \( A \).

(75) For every symmetrical subset \( A \) of \( \mathbb{R} \) holds \( \text{(the function sin)}^{-1} \) is odd on \( A \).

(76) For every symmetrical subset \( A \) of \( \mathbb{R} \) holds \( \text{(the function cos)}^{-1} \) is even on \( A \).

(77) For every symmetrical subset \( A \) of \( \mathbb{R} \) holds \(-\text{the function sin} \) is odd on \( A \).

(78) For every symmetrical subset \( A \) of \( \mathbb{R} \) holds \(-\text{the function cos} \) is even on \( A \).

(79) For every symmetrical subset \( A \) of \( \mathbb{R} \) holds \( \text{(the function sin)}^2 \) is even on \( A \).

(80) For every symmetrical subset \( A \) of \( \mathbb{R} \) holds \( \text{(the function cos)}^2 \) is even on \( A \).
In the sequel $B$ is a symmetrical subset of $\mathbb{R}$.

One can prove the following propositions:

(81) If $B \subseteq \text{dom (the function sec)}$, then the function sec is even on $B$.

(82) If for every real number $x$ such that $x \in B$ holds (the function cos)$(x) \neq 0$, then the function sec is even on $B$.

(83) If $B \subseteq \text{dom (the function cosec)}$, then the function cosec is odd on $B$.

(84) If for every real number $x$ such that $x \in B$ holds (the function sin)$(x) \neq 0$, then the function cosec is odd on $B$.

References


Received May 25, 2009