Collective Operations on Number-Membered Sets

Artur Korniłowicz
Institute of Computer Science
University of Białystok
Sosnowa 64, 15-887 Białystok
Poland

Summary. The article starts with definitions of sets of opposite and inverse numbers of a given number membered set. Next, collective addition, subtraction, multiplication and division of two sets are defined. Complex numbers cases and extended real numbers ones are introduced separately and unified for reals. Shortcuts for singletons cases are also defined.

MML identifier: MEMBER_1, version: 7.11.01 4.117.1046

The articles [4], [2], [1], and [3] provide the notation and terminology for this paper.

For simplicity, we use the following convention: \( w, w_1, w_2 \) denote elements of \( \mathbb{R} \), \( c, c_1, c_2 \) denote elements of \( \mathbb{C} \), \( A, B, C, D \) denote complex-membered sets, \( F, G, H, I \) denote extended real-membered sets, \( a, b \) denote complex numbers, \( f, g \) denote extended real numbers, \( r \) denotes a real number, and \( e \) denotes a set.

Let us consider \( w \). Then \( -w \) is an element of \( \mathbb{R} \). Then \( w^{-1} \) is an element of \( \mathbb{R} \).

Let \( a, b, c, d \) be complex numbers. Note that \( \{ a, b, c, d \} \) is complex-membered.

Let \( a, b, c, d \) be extended real numbers. Observe that \( \{ a, b, c, d \} \) is extended real-membered.

Let \( F \) be an extended real-membered set. The functor \( \ominus F \) yielding an extended real-membered set is defined as follows:

(Def. 1) \( \ominus F = \{ -w : w \in F \} \).
Let us observe that the functor $\ominus F$ is involutive.

Next we state two propositions:

(1) $f \in F$ iff $-f \in \ominus F$.

(2) $-f \in F$ iff $f \in \ominus F$.

Let $F$ be an empty set. Observe that $\ominus F$ is empty.

Let $F$ be an extended real-membered non empty set. One can check that $\ominus F$ is non empty.

One can prove the following propositions:

(3) $F \subseteq G$ iff $\ominus F \subseteq \ominus G$.

(4) If $\ominus F = \ominus G$, then $F = G$.

(5) $\ominus(F \cup G) = \ominus F \cup \ominus G$.

(6) $\ominus(F \cap G) = \ominus F \cap \ominus G$.

(7) $\ominus(F \setminus G) = \ominus F \setminus \ominus G$.

(8) $\ominus(F \vee G) = \ominus F \vee \ominus G$.

(9) $\ominus\{f\} = \{-f\}$.

(10) $\ominus\{f, g\} = \{-f, -g\}$.

Let $A$ be a complex-membered set. The functor $\ominus A$ yields a complex-membered set and is defined as follows:

(Def. 2) $\ominus A = \{-c : c \in A\}$.

Let us notice that the functor $\ominus A$ is involutive.

We now state two propositions:

(11) $a \in A$ iff $-a \in \ominus A$.

(12) $-a \in A$ iff $a \in \ominus A$.

Let $A$ be an empty set. One can check that $\ominus A$ is empty.

Let $A$ be a complex-membered non empty set. One can verify that $\ominus A$ is non empty.

Let $A$ be a real-membered set. Note that $\ominus A$ is real-membered.

Let $A$ be a rational-membered set. One can check that $\ominus A$ is rational-membered.

Let $A$ be an integer-membered set. Observe that $\ominus A$ is integer-membered.

Next we state several propositions:

(13) $A \subseteq B$ iff $\ominus A \subseteq \ominus B$.

(14) If $\ominus A = \ominus B$, then $A = B$.

(15) $\ominus(A \cup B) = \ominus A \cup \ominus B$.

(16) $\ominus(A \cap B) = \ominus A \cap \ominus B$.

(17) $\ominus(A \setminus B) = \ominus A \setminus \ominus B$.

(18) $\ominus(A \vee B) = \ominus A \vee \ominus B$.

(19) $\ominus\{a\} = \{-a\}$.
Let $F$ be an extended real-membered set. The functor $F^{-1}$ yielding an extended real-membered set is defined by:

\[(\text{Def. 3}) \quad F^{-1} = \{w^{-1} : w \in F\}.\]

We now state the proposition

\[(21) \quad \text{If } f \in F, \text{ then } f^{-1} \in F^{-1}.\]

Let $F$ be an empty set. Observe that $F^{-1}$ is empty.

Let $F$ be an extended real-membered non empty set. One can verify that $F^{-1}$ is non empty.

One can prove the following propositions:

\[(22) \quad \text{If } F \subseteq G, \text{ then } F^{-1} \subseteq G^{-1}.\]

\[(23) \quad (F \cup G)^{-1} = F^{-1} \cup G^{-1}.\]

\[(24) \quad (F \cap G)^{-1} \subseteq F^{-1} \cap G^{-1}.\]

\[(25) \quad \ominus(F^{-1}) = (\ominus F)^{-1}.\]

\[(26) \quad \{f\}^{-1} = \{f^{-1}\}.\]

\[(27) \quad \{f, g\}^{-1} = \{f^{-1}, g^{-1}\}.\]

Let $A$ be a complex-membered set. The functor $A^{-1}$ yields a complex-membered set and is defined by:

\[(\text{Def. 4}) \quad A^{-1} = \{c^{-1} : c \in A\}.\]

Let us notice that the functor $A^{-1}$ is involutive.

We now state two propositions:

\[(28) \quad a \in A \text{ iff } a^{-1} \in A^{-1}.\]

\[(29) \quad a^{-1} \in A \text{ iff } a \in A^{-1}.\]

Let $A$ be an empty set. Note that $A^{-1}$ is empty.

Let $A$ be a complex-membered non empty set. Note that $A^{-1}$ is non empty.

Let $A$ be a real-membered set. Observe that $A^{-1}$ is real-membered.

Let $A$ be a rational-membered set. One can check that $A^{-1}$ is rational-membered.

Next we state several propositions:

\[(30) \quad A \subseteq B \text{ iff } A^{-1} \subseteq B^{-1}.\]

\[(31) \quad \text{If } A^{-1} = B^{-1}, \text{ then } A = B.\]

\[(32) \quad (A \cup B)^{-1} = A^{-1} \cup B^{-1}.\]

\[(33) \quad (A \cap B)^{-1} = A^{-1} \cap B^{-1}.\]

\[(34) \quad (A \setminus B)^{-1} = A^{-1} \setminus B^{-1}.\]

\[(35) \quad (A \ominus B)^{-1} = A^{-1} \ominus B^{-1}.\]

\[(36) \quad \ominus(A^{-1}) = (\ominus A)^{-1}.\]

\[(37) \quad \{a\}^{-1} = \{a^{-1}\}.\]

\[(38) \quad \{a, b\}^{-1} = \{a^{-1}, b^{-1}\}.\]
Let $F$, $G$ be extended real-membered sets. The functor $F \oplus G$ is defined by:

\[(\text{Def. 5}) \quad F \oplus G = \{ w_1 + w_2 : w_1 \in F \land w_2 \in G \}.\]

Let us observe that the functor $F \oplus G$ is commutative.

Next we state the proposition

\[(39) \quad \text{If } f \in F \text{ and } g \in G, \text{ then } f + g \in F \oplus G.\]

Let $F$ be an empty set and let $G$ be an extended real-membered set. Note that $F \oplus G$ is empty and $G \oplus F$ is empty.

Let $F$, $G$ be extended real-membered non empty sets. Observe that $F \oplus G$ is non empty.

Let $F$ be an empty set and let $G$ be an extended real-membered set. Note that $F \oplus G$ is empty and $G \oplus F$ is empty.

Let $F$, $G$ be extended real-membered sets. Observe that $F \oplus G$ is extended real-membered.

One can prove the following propositions:

\[(40) \quad \text{If } F \subseteq G \text{ and } H \subseteq I, \text{ then } F \oplus H \subseteq G \oplus I.\]

\[(41) \quad F \oplus (G \cup H) = (F \oplus G) \cup (F \oplus H).\]

\[(42) \quad F \oplus G \cap H \subseteq (F \oplus G) \cap (F \oplus H).\]

\[(43) \quad \{ f \} \oplus \{ g \} = \{ f + g \}.\]

\[(44) \quad \{ f \} \oplus \{ g, h \} = \{ f + g, f + h \}.\]

\[(45) \quad \{ f, g \} \oplus \{ h, i \} = \{ f + h, f + i, g + h, g + i \}.\]

Let $A$, $B$ be complex-membered sets. The functor $A \oplus B$ is defined by:

\[(\text{Def. 6}) \quad A \oplus B = \{ c_1 + c_2 : c_1 \in A \land c_2 \in B \}.\]

Let us note that the functor $A \oplus B$ is commutative.

Next we state the proposition

\[(46) \quad \text{If } a \in A \text{ and } b \in B, \text{ then } a + b \in A \oplus B.\]

Let $A$ be an empty set and let $B$ be a complex-membered set. One can verify that $A \oplus B$ is empty and $B \oplus A$ is empty.

Let $A$, $B$ be complex-membered non empty sets. Note that $A \oplus B$ is non empty.

Let $A$, $B$ be complex-membered sets. Observe that $A \oplus B$ is complex-membered.

Let $A$, $B$ be real-membered sets. One can check that $A \oplus B$ is real-membered.

Let $A$, $B$ be rational-membered sets. Observe that $A \oplus B$ is rational-membered.

Let $A$, $B$ be integer-membered sets. Observe that $A \oplus B$ is integer-membered.

Let $A$, $B$ be natural-membered sets. Note that $A \oplus B$ is natural-membered.

We now state several propositions:

\[(47) \quad \text{If } A \subseteq B \text{ and } C \subseteq D, \text{ then } A \oplus C \subseteq B \oplus D.\]

\[(48) \quad A \oplus (B \cup C) = (A \oplus B) \cup (A \oplus C).\]

\[(49) \quad A \oplus B \cap C \subseteq (A \oplus B) \cap (A \oplus C).\]

\[(50) \quad (A \oplus B) \oplus C = A \oplus (B \oplus C).\]

\[(51) \quad \{ a \} \oplus \{ b \} = \{ a + b \}.\]
collective operations on number-membered sets

(52) \( \{a\} \oplus \{s, t\} = \{a + s, a + t\} \).

(53) \( \{a, b\} \oplus \{s, t\} = \{a + s, a + t, b + s, b + t\} \).

Let \( F, G \) be extended real-membered sets. The functor \( F \otimes G \) is defined by:

(Def. 7) \( F \otimes G = F \oplus \ominus G \).

We now state two propositions:

(54) \( F \otimes G = \{w_1 - w_2 : w_1 \in F \land w_2 \in G\} \).

(55) If \( f \in F \) and \( g \in G \), then \( f - g \in F \otimes G \).

Let \( F \) be an empty set and let \( G \) be an extended real-membered set. One can verify that \( F \otimes G \) is empty and \( G \otimes F \) is empty.

Let \( F, G \) be extended real-membered non empty sets. Observe that \( F \otimes G \) is non empty.

Let \( F, G \) be extended real-membered sets. Note that \( F \otimes G \) is extended real-membered.

Next we state several propositions:

(56) If \( F \subseteq G \) and \( H \subseteq I \), then \( F \otimes H \subseteq G \otimes I \).

(57) \( F \otimes (G \cup H) = (F \otimes G) \cup (F \otimes H) \).

(58) \( F \otimes G \cap H \subseteq (F \otimes G) \cap (F \otimes H) \).

(59) \( \ominus (F \oplus G) = \ominus F \oplus G \).

(60) \( \ominus (F \otimes G) = \ominus F \otimes G \).

(61) \( \{f\} \ominus \{g\} = \{f - g\} \).

(62) \( \{f\} \ominus \{h, i\} = \{f - h, f - i\} \).

(63) \( \{f, g\} \ominus \{h\} = \{f - h, g - h\} \).

(64) \( \{f, g\} \ominus \{h, i\} = \{f - h, f - i, g - h, g - i\} \).

Let \( A, B \) be complex-membered sets. The functor \( A \ominus B \) is defined as follows:

(Def. 8) \( A \ominus B = A \oplus \ominus B \).

We now state two propositions:

(65) \( A \ominus B = \{c_1 - c_2 : c_1 \in A \land c_2 \in B\} \).

(66) If \( a \in A \) and \( b \in B \), then \( a - b \in A \ominus B \).

Let \( A \) be an empty set and let \( B \) be a complex-membered set. Observe that \( A \ominus B \) is empty and \( B \ominus A \) is empty.

Let \( A, B \) be complex-membered non empty sets. Observe that \( A \ominus B \) is non empty.

Let \( A, B \) be complex-membered sets. One can check that \( A \ominus B \) is complex-membered.

Let \( A, B \) be real-membered sets. Note that \( A \ominus B \) is real-membered.

Let \( A, B \) be rational-membered sets. Observe that \( A \ominus B \) is rational-membered.

Let \( A, B \) be integer-membered sets. Observe that \( A \ominus B \) is integer-membered.

One can prove the following propositions:
102

(67) If \( A \subseteq B \) and \( C \subseteq D \), then \( A \odot C \subseteq B \odot D \).

(68) \( A \ominus (B \cup C) = (A \ominus B) \cup (A \ominus C) \).

(69) \( A \ominus B \cap C \subseteq (A \ominus B) \cap (A \ominus C) \).

(70) \( \ominus (A \oplus B) = \ominus A \ominus B \).

(71) \( \ominus (A \ominus B) = \ominus A \ominus B \).

(72) \( A \ominus (B \ominus C) = (A \ominus B) \ominus C \).

(73) \( A \ominus (B \ominus C) = A \ominus B \ominus C \).

(74) \( A \ominus (A \ominus B) \ominus C \).

(75) \( \{a\} \ominus \{b\} = \{a - b\} \).

(76) \( \{a\} \ominus \{s, t\} = \{a - s, a - t\} \).

(77) \( \{a, b\} \ominus \{s\} = \{a - s, b - s\} \).

(78) \( \{a, b\} \ominus \{s, t\} = \{a - s, a - t, b - s, b - t\} \).

Let \( F, G \) be extended real-membered sets. The functor \( F \circ G \) is defined by:

(Def. 9) \( F \circ G = \{w_1 \cdot w_2 : w_1 \in F \land w_2 \in G\} \).

Let us observe that the functor \( F \circ G \) is commutative.

Let \( F \) be an empty set and let \( G \) be an extended real-membered set. Note that \( F \circ G \) is empty and \( G \circ F \) is empty.

Let \( F, G \) be extended real-membered sets. Observe that \( F \circ G \) is extended real-membered.

One can prove the following proposition

(79) If \( f \in F \) and \( g \in G \), then \( f \cdot g \in F \circ G \).

Let \( F, G \) be extended real-membered non empty sets. Note that \( F \circ G \) is non empty.

Next we state several propositions:

(80) \( (F \circ G) \circ H = F \circ (G \circ H) \).

(81) If \( F \subseteq G \) and \( H \subseteq I \), then \( F \circ H \subseteq G \circ I \).

(82) \( F \circ (G \cup H) = F \circ G \cup F \circ H \).

(83) \( F \circ (G \cap H) \subseteq (F \circ G) \cap (F \circ H) \).

(84) \( F \circ G = \ominus (F \circ G) \).

(85) \( (F \circ G)^{-1} = F^{-1} \circ G^{-1} \).

(86) \( \{f\} \circ \{g\} = \{f \cdot g\} \).

(87) \( \{f\} \circ \{h, i\} = \{f \cdot h, f \cdot i\} \).

(88) \( \{f, g\} \circ \{h, i\} = \{f \cdot h, f \cdot i, g \cdot h, g \cdot i\} \).

Let \( A, B \) be complex-membered sets. The functor \( A \circ B \) is defined by:

(Def. 10) \( A \circ B = \{c_1 \cdot c_2 : c_1 \in A \land c_2 \in B\} \).

Let us notice that the functor \( A \circ B \) is commutative.

We now state the proposition

(89) If \( a \in A \) and \( b \in B \), then \( a \cdot b \in A \circ B \).
Let $A$ be an empty set and let $B$ be a complex-membered set. Note that $A \circ B$ is empty and $B \circ A$ is empty.

Let $A, B$ be complex-membered non empty sets. Observe that $A \circ B$ is non empty.

Let $A, B$ be complex-membered sets. Observe that $A \circ B$ is complex-membered.

Let $A, B$ be real-membered sets. One can verify that $A \circ B$ is real-membered.

Let $A, B$ be rational-membered sets. Observe that $A \circ B$ is rational-membered.

Let $A, B$ be integer-membered sets. Observe that $A \circ B$ is integer-membered.

Let $A, B$ be natural-membered sets. Observe that $A \circ B$ is natural-membered.

One can prove the following propositions:

(90) $(A \circ B) \circ C = A \circ (B \circ C)$.

(91) If $A \subseteq B$ and $C \subseteq D$, then $A \circ C \subseteq B \circ D$.

(92) $A \circ (B \cup C) = A \circ B \cup A \circ C$.

(93) $A \circ (B \cap C) \subseteq (A \circ B) \cap (A \circ C)$.

(94) $A \circ \circ B = \circ (A \circ B)$.

(95) $A \circ (B \oplus C) \subseteq A \circ B \oplus A \circ C$.

(96) $A \circ (B \ominus C) \subseteq A \circ B \ominus A \circ C$.

(97) $(A \circ B)^{-1} = A^{-1} \circ B^{-1}$.

(98) $\{a\} \circ \{b\} = \{a \cdot b\}$.

(99) $\{a\} \circ \{s, t\} = \{a \cdot s, a \cdot t\}$.

(100) $\{a, b\} \circ \{s, t\} = \{a \cdot s, a \cdot t, b \cdot s, b \cdot t\}$.

Let $F, G$ be extended real-membered sets. The functor $F \otimes G$ is defined as follows:

(Def. 11) $F \otimes G = F \circ G^{-1}$.

We now state two propositions:

(101) $F \otimes G = \{\frac{w_1}{w_2} : w_1 \in F \land w_2 \in G\}$.

(102) If $f \in F$ and $g \in G$, then $\frac{f}{g} \in F \otimes G$.

Let $F$ be an empty set and let $G$ be an extended real-membered set. One can check that $F \otimes G$ is empty and $G \otimes F$ is empty.

Let $F, G$ be extended real-membered non empty sets. One can check that $F \otimes G$ is non empty.

Let $F, G$ be extended real-membered sets. Observe that $F \otimes G$ is extended real-membered.

One can prove the following propositions:

(103) If $F \subseteq G$ and $H \subseteq I$, then $F \otimes H \subseteq G \circ I$.

(104) $(F \cup G) \circ H = (F \circ H) \cup (G \circ H)$.

(105) $F \cap G \circ H \subseteq (F \circ H) \cap (G \circ H)$.

(106) $F \circ (G \cup H) = (F \circ G) \cup (F \circ H)$. 
\[ F \odot G \cap H \subseteq (F \odot G) \cap (F \odot H). \]

\[ F \circ G \circ H = F \circ (G \odot H). \]

\[ (F \odot G) \circ H = F \circ H \odot G. \]

\[ F \odot G \circ H = F \odot G \circ H. \]

\[ \{f\} \odot \{g\} = \{\frac{f}{g}\}. \]

\[ \{f\} \odot \{h, i\} = \{\frac{f}{h}, \frac{f}{i}\}. \]

\[ \{f, g\} \odot \{h\} = \{\frac{f}{h}, \frac{g}{h}\}. \]

\[ \{f, g\} \odot \{h, i\} = \{\frac{f}{h}, \frac{f}{i}, \frac{g}{h}, \frac{g}{i}\}. \]

Let \( A, B \) be complex-membered sets. The functor \( A \odot B \) is defined as follows:

(Def. 12) \[ A \odot B = A \circ B^{-1}. \]

One can prove the following two propositions:

(115) \[ A \odot B = \{\frac{c_1}{c_2} : c_1 \in A \land c_2 \in B\}. \]

(116) If \( a \in A \) and \( b \in B \), then \( \frac{a}{b} \in A \odot B \).

Let \( A \) be an empty set and let \( B \) be a complex-membered set. One can check that \( A \odot B \) is empty and \( B \odot A \) is empty.

Let \( A, B \) be complex-membered non empty sets. Note that \( A \odot B \) is non empty.

Let \( A, B \) be complex-membered sets. Note that \( A \odot B \) is complex-membered.

Let \( A, B \) be real-membered sets. Note that \( A \odot B \) is real-membered.

Let \( A, B \) be rational-membered sets. Note that \( A \odot B \) is rational-membered.

We now state a number of propositions:

(117) If \( A \subseteq B \) and \( C \subseteq D \), then \( A \odot C \subseteq B \odot D \).

(118) \[ A \odot (B \cup C) = (A \odot B) \cup (A \odot C). \]

(119) \[ A \odot B \cap C \subseteq (A \odot B) \cap (A \odot C). \]

(120) \[ A \odot B = \ominus(A \odot B). \]

(121) \[ \ominus A \odot B = \ominus(A \odot B). \]

(122) \[ (A \odot B) \odot C \subseteq (A \odot C) \oplus (B \odot C). \]

(123) \[ (A \odot B) \odot C \subseteq (A \odot C) \oplus (B \odot C). \]

(124) \[ A \circ B \circ C = A \circ (B \circ C). \]

(125) \[ A \circ B \circ C = A \circ C \circ B. \]

(126) \[ A \circ B \circ C = A \circ B \circ C. \]

(127) \[ A \circ (B \odot C) = A \circ C \odot B. \]

(128) \[ \{a\} \odot \{b\} = \{\frac{a}{b}\}. \]

(129) \[ \{a\} \odot \{s, t\} = \{\frac{a}{s}, \frac{a}{t}\}. \]

(130) \[ \{a, b\} \odot \{s\} = \{\frac{a}{s}, \frac{b}{s}\}. \]

(131) \[ \{a, b\} \odot \{s, t\} = \{\frac{a}{s}, \frac{a}{t}, \frac{b}{s}, \frac{b}{t}\}. \]
Let $F$ be an extended real-membered set and let $f$ be an extended real number. The functor $f ⊕ F$ is defined by:

(Def. 13) $f ⊕ F = \{ f \} ⊕ F$.

One can prove the following three propositions:

(132) If $g ∈ G$, then $f + g ∈ f ⊕ G$.
(133) $f ⊕ F = \{ f + w : w ∈ F \}$.
(134) If $e ∈ f ⊕ F$, then there exists $w$ such that $e = f + w$ and $w ∈ F$.

Let $F$ be an empty set and let $f$ be an extended real number. One can verify that $f ⊕ F$ is empty.

Let $F$ be an extended real-membered non empty set and let $f$ be an extended real number. Note that $f ⊕ F$ is non empty.

Let $F$ be an extended real-membered set and let $f$ be an extended real number. Observe that $f ⊕ F$ is extended real-membered.

One can prove the following propositions:

(135) If $r ⊕ F ⊆ r ⊕ G$, then $F ⊆ G$.
(136) If $r ⊕ F = r ⊕ G$, then $F = G$.
(137) $r ⊕ F ∩ G = (r ⊕ F) ∩ (r ⊕ G)$.
(138) $(f ⊕ F) \setminus (f ⊕ G) ⊆ f ⊕ (F \setminus G)$.
(139) $r ⊕ (F \setminus G) = (r ⊕ F) \setminus (r ⊕ G)$.
(140) $r ⊕ (F \setminus G) = (r ⊕ F) \setminus (r ⊕ G)$.

Let $A$ be a complex-membered set and let $a$ be a complex number. The functor $a ⊕ A$ is defined by:

(Def. 14) $a ⊕ A = \{ a \} ⊕ A$.

One can prove the following propositions:

(141) If $b ∈ A$, then $a + b ∈ a ⊕ A$.
(142) $a ⊕ A = \{ a + c : c ∈ A \}$.
(143) If $e ∈ a ⊕ A$, then there exists $c$ such that $e = a + c$ and $c ∈ A$.

Let $A$ be an empty set and let $a$ be a complex number. Observe that $a ⊕ A$ is empty.

Let $A$ be a complex-membered non empty set and let $a$ be a complex number. One can verify that $a ⊕ A$ is non empty.

Let $A$ be a complex-membered set and let $a$ be a complex number. Observe that $a ⊕ A$ is complex-membered.

Let $A$ be a real-membered set and let $a$ be a real number. Observe that $a ⊕ A$ is real-membered.

Let $A$ be a rational-membered set and let $a$ be a rational number. Note that $a ⊕ A$ is rational-membered.

Let $A$ be an integer-membered set and let $a$ be an integer number. Observe that $a ⊕ A$ is integer-membered.
Let $A$ be a natural-membered set and let $a$ be a natural number. Note that $a \oplus A$ is natural-membered.

We now state several propositions:

(144) $A \subseteq B$ iff $a \oplus A \subseteq a \oplus B$.

(145) If $a \oplus A = a \oplus B$, then $A = B$.

(146) $0 \oplus A = A$.

(147) $(a + b) \oplus A = a \oplus (b \oplus A)$.

(148) $a \oplus (A \oplus B) = (a \oplus A) \oplus B$.

(149) $a \oplus A \cap B = (a \oplus A) \cap (a \oplus B)$.

(150) $a \oplus (A \setminus B) = (a \oplus A) \setminus (a \oplus B)$.

(151) $a \oplus (A \setminus \cdot B) = (a \oplus A) \setminus (a \oplus B)$.

Let $F$ be an extended real-membered set and let $f$ be an extended real number. The functor $f \ominus F$ is defined by:

(Def. 15) $f \ominus F = \{f\} \ominus F$.

Next we state three propositions:

(152) If $g \in G$, then $f - g \in f \ominus G$.

(153) $f \ominus F = \{f - w : w \in F\}$.

(154) If $e \in f \ominus F$, then there exists $w$ such that $e = f - w$ and $w \in F$.

Let $F$ be an empty set and let $f$ be an extended real number. Note that $f \ominus F$ is empty.

Let $F$ be an extended real-membered non empty set and let $f$ be an extended real number. Note that $f \ominus F$ is non empty.

Let $F$ be an extended real-membered set and let $f$ be an extended real number. Note that $f \ominus F$ is extended real-membered.

One can prove the following propositions:

(155) If $r \ominus F \subseteq r \ominus G$, then $F \subseteq G$.

(156) If $r \ominus F = r \ominus G$, then $F = G$.

(157) $r \ominus F \cap G = (r \ominus F) \cap (r \ominus G)$.

(158) $r \ominus (F \setminus G) = (r \ominus F) \setminus (r \ominus G)$.

(159) $r \ominus (F \setminus \cdot G) = (r \ominus F) \setminus (r \ominus G)$.

Let $A$ be a complex-membered set and let $a$ be a complex number. The functor $a \ominus A$ is defined as follows:

(Def. 16) $a \ominus A = \{a\} \ominus A$.

Next we state three propositions:

(160) If $b \in A$, then $a - b \in a \ominus A$.

(161) $a \ominus A = \{a - c : c \in A\}$.

(162) If $e \in a \ominus A$, then there exists $c$ such that $e = a - c$ and $c \in A$. 
Let $A$ be an empty set and let $a$ be a complex number. One can check that $a \ominus A$ is empty.

Let $A$ be a complex-membered non empty set and let $a$ be a complex number. One can check that $a \ominus A$ is non empty.

Let $A$ be a complex-membered set and let $a$ be a complex number. One can check that $a \ominus A$ is complex-membered.

Let $A$ be a real-membered set and let $a$ be a real number. Observe that $a \ominus A$ is real-membered.

Let $A$ be a rational-membered set and let $a$ be a rational number. Note that $a \ominus A$ is rational-membered.

Let $A$ be an integer-membered set and let $a$ be an integer number. One can verify that $a \ominus A$ is integer-membered.

One can prove the following propositions:

\begin{enumerate}
\item[(163)] $A \subseteq B$ iff $a \ominus A \subseteq a \ominus B$.
\item[(164)] If $a \ominus A = a \ominus B$, then $A = B$.
\item[(165)] $a \ominus A \cap B = (a \ominus A) \cap (a \ominus B)$.
\item[(166)] $a \ominus (A \setminus B) = (a \ominus A) \setminus (a \ominus B)$.
\item[(167)] $a \ominus (A \setminus B) = (a \ominus A) \setminus (a \ominus B)$.
\end{enumerate}

Let $F$ be an extended real-membered set and let $f$ be an extended real number. The functor $F \ominus f$ is defined by:

(Def. 17) $F \ominus f = F \ominus \{f\}$.

Next we state three propositions:

\begin{enumerate}
\item[(168)] If $g \in G$, then $g - f \in G \ominus f$.
\item[(169)] $F \ominus f = \{w - f : w \in F\}$.
\item[(170)] If $e \in F \ominus f$, then there exists $w$ such that $e = w - f$ and $w \in F$.
\end{enumerate}

Let $F$ be an empty set and let $f$ be an extended real number. One can check that $F \ominus f$ is empty.

Let $F$ be an extended real-membered non empty set and let $f$ be an extended real number. One can check that $F \ominus f$ is non empty.

Let $F$ be an extended real-membered set and let $f$ be an extended real number. One can check that $F \ominus f$ is extended real-membered.

Next we state several propositions:

\begin{enumerate}
\item[(171)] $F \ominus f = \ominus(f \ominus F)$.
\item[(172)] $f \ominus F = \ominus(F \ominus f)$.
\item[(173)] $F \cap G \ominus r = (F \ominus r) \cap (G \ominus r)$.
\item[(174)] $(F \setminus G) \ominus r = (F \ominus r) \setminus (G \ominus r)$.
\item[(175)] $(F \setminus G) \ominus r = (F \ominus r) \setminus (G \ominus r)$.
\end{enumerate}

Let $A$ be a complex-membered set and let $a$ be a complex number. The functor $A \ominus a$ is defined as follows:
(Def. 18) \[ A \ominus a = A \ominus \{a\}. \]

One can prove the following three propositions:

176) If \( b \in A \), then \( b - a \in A \ominus a \).

177) \[ A \ominus a = \{c - a : c \in A\}. \]

178) If \( e \in A \ominus a \), then there exists \( c \) such that \( e = c - a \) and \( c \in A \).

Let \( A \) be an empty set and let \( a \) be a complex number. Note that \( A \ominus a \) is empty.

Let \( A \) be a complex-membered non empty set and let \( a \) be a complex number. Observe that \( A \ominus a \) is complex-membered.

Let \( A \) be a complex-membered set and let \( a \) be a complex number. Observe that \( A \ominus a \) is complex-membered.

Let \( A \) be an integer-membered set and let \( a \) be an integer number. One can check that \( A \ominus a \) is integer-membered.

One can prove the following propositions:

179) \( A \subseteq B \) iff \( A \ominus a \subseteq B \ominus a \).

180) If \( A \ominus a = B \ominus a \), then \( A = B \).

181) \[ A \ominus a = \ominus(a \ominus A). \]

182) \[ a \ominus A = \ominus(A \ominus a). \]

183) \[ A \cap B \ominus a = (A \ominus a) \cap (B \ominus a). \]

184) \[ (A \setminus B) \ominus a = (A \ominus a) \setminus (B \ominus a). \]

185) \[ (A - B) \ominus a = (A \ominus a) - (B \ominus a). \]

Let \( F \) be an extended real-membered set and let \( f \) be an extended real number. The functor \( f \circ F \) is defined as follows:

(Def. 19) \[ f \circ F = \{f\} \circ F. \]

The following propositions are true:

186) If \( g \in G \), then \( f \cdot g \in f \circ G \).

187) \[ f \circ F = \{f \cdot w : w \in F\}. \]

188) If \( e \in f \circ F \), then there exists \( w \) such that \( e = f \cdot w \) and \( w \in F \).

Let \( F \) be an empty set and let \( f \) be an extended real number. Note that \( f \circ F \) is empty.

Let \( F \) be an extended real-membered non empty set and let \( f \) be an extended real number. Note that \( f \circ F \) is non empty.

Let \( F \) be an extended real-membered set and let \( f \) be an extended real number. Observe that \( f \circ F \) is extended real-membered.

One can prove the following four propositions:
If \( r \neq 0 \), then \( r \circ (F \cap G) = (r \circ F) \cap (r \circ G) \).

(190) If \( r \neq 0 \), then \( f \circ F \setminus f \circ G \subseteq f \circ (F \setminus G) \).

(191) If \( r \neq 0 \), then \( r \circ (F \setminus G) = r \circ F \setminus r \circ G \).

(192) If \( r \neq 0 \), then \( r \circ (F - G) = r \circ F - r \circ G \).

Let \( A \) be a complex-membered set and let \( a \) be a complex number. The functor \( a \circ A \) is defined by:

(Def. 20) \( a \circ A = \{a\} \circ A \).

Next we state three propositions:

(193) If \( b \in A \), then \( a \cdot b \in a \circ A \).

(194) \( a \circ A = \{a \cdot c : c \in A\} \).

(195) If \( e \in a \circ A \), then there exists \( c \) such that \( e = a \cdot c \) and \( c \in A \).

Let \( A \) be an empty set and let \( a \) be a complex number. Note that \( a \circ A \) is empty.

Let \( A \) be a complex-membered non empty set and let \( a \) be a complex number. Observe that \( a \circ A \) is non empty.

Let \( A \) be a complex-membered set and let \( a \) be a complex number. Observe that \( a \circ A \) is complex-membered.

Let \( A \) be a real-membered set and let \( a \) be a real number. One can check that \( a \circ A \) is real-membered.

Let \( A \) be a rational-membered set and let \( a \) be a rational number. Note that \( a \circ A \) is rational-membered.

Let \( A \) be an integer-membered set and let \( a \) be an integer number. Observe that \( a \circ A \) is integer-membered.

Let \( A \) be a natural-membered set and let \( a \) be a natural number. Note that \( a \circ A \) is natural-membered.

We now state a number of propositions:

(196) If \( a \neq 0 \) and \( a \circ A \subseteq a \circ B \), then \( A \subseteq B \).

(197) If \( a \neq 0 \) and \( a \circ A = a \circ B \), then \( A = B \).

(198) If \( a \neq 0 \), then \( a \circ (A \cap B) = (a \circ A) \cap (a \circ B) \).

(199) If \( a \neq 0 \), then \( a \circ (A \setminus B) = a \circ A \setminus a \circ B \).

(200) If \( a \neq 0 \), then \( a \circ (A - B) = a \circ A - a \circ B \).

(201) \( 0 \circ A \subseteq \{0\} \).

(202) If \( A \neq 0 \), then \( 0 \circ A = \{0\} \).

(203) \( 1 \circ A = A \).

(204) \( (a \cdot b) \circ A = a \circ (b \circ A) \).

(205) \( a \circ (A \circ B) = (a \circ A) \circ B \).

(206) \( (a + b) \circ A \subseteq a \circ A \oplus b \circ A \).

(207) \( (a - b) \circ A \subseteq a \circ A \ominus b \circ A \).
Let $F$ be an extended real-membered set and let $f$ be an extended real number. The functor $f \odot F$ is defined as follows:

(Def. 21) \[ f \odot F = \{f\} \odot F. \]

The following three propositions are true:

(208) \[ a \odot (B \odot C) = a \odot B \oplus a \odot C. \]

(209) \[ a \odot (B \odot C) = a \odot B \odot a \odot C. \]

(210) \[ f \odot G, \text{ then } \frac{f}{g} \in f \odot G. \]

(211) \[ f \odot F = \{ \frac{f}{w} : w \in F \}. \]

(212) \[ e \in f \odot F, \text{ then there exists } w \text{ such that } e = \frac{f}{w} \text{ and } w \in F. \]

Let $F$ be an empty set and let $f$ be an extended real number. Observe that $f \odot F$ is empty.

Let $F$ be an extended real-membered non empty set and let $f$ be an extended real number. Observe that $f \odot F$ is non empty.

Let $F$ be an extended real-membered set and let $f$ be an extended real number. Observe that $f \odot F$ is extended real-membered.

Let $A$ be a complex-membered set and let $a$ be a complex number. The functor $a \odot A$ is defined by:

(Def. 22) \[ a \odot A = \{a\} \odot A. \]

Next we state three propositions:

(213) \[ b \in A, \text{ then } \frac{a}{b} \in a \odot A. \]

(214) \[ a \odot A = \{ \frac{a}{c} : c \in A \}. \]

(215) \[ e \in a \odot A, \text{ then there exists } c \text{ such that } e = \frac{a}{c} \text{ and } c \in A. \]

Let $A$ be an empty set and let $a$ be a complex number. Observe that $a \odot A$ is empty.

Let $A$ be a complex-membered non empty set and let $a$ be a complex number. Note that $a \odot A$ is non empty.

Let $A$ be a complex-membered set and let $a$ be a complex number. Note that $a \odot A$ is complex-membered.

Let $A$ be a real-membered set and let $a$ be a real number. Note that $a \odot A$ is real-membered.

Let $A$ be a rational-membered set and let $a$ be a rational number. One can verify that $a \odot A$ is rational-membered.

Next we state several propositions:

(216) \[ a \neq 0 \text{ and } a \odot A \subseteq a \odot B, \text{ then } A \subseteq B. \]

(217) \[ a \neq 0 \text{ and } a \odot A = a \odot B, \text{ then } A = B. \]

(218) \[ a \neq 0, \text{ then } a \odot A \cap B = (a \odot A) \cap (a \odot B). \]

(219) \[ a \neq 0, \text{ then } a \odot (A \setminus B) = (a \odot A) \setminus (a \odot B). \]

(220) \[ a \neq 0, \text{ then } a \odot (A \setminus B) = (a \odot A) \setminus (a \odot B). \]

(221) \[ (a + b) \odot A \subseteq (a \odot A) \oplus (b \odot A). \]
Let $F$ be an extended real-membered set and let $f$ be an extended real number. The functor $F \odot f$ is defined as follows:

(Def. 23) $F \odot f = F \odot \{f\}$.

One can prove the following propositions:

(223) If $g \in G$, then $g f \in G \odot f$.

(224) $F \odot f = \{ w f : w \in F \}$.

(225) If $e \in F \odot f$, then there exists $w$ such that $e = \frac{w}{f}$ and $w \in F$.

Let $F$ be an empty set and let $f$ be an extended real number. One can check that $F \odot f$ is empty.

Let $F$ be an extended real-membered non empty set and let $f$ be an extended real number. One can check that $F \odot f$ is extended real-membered.

Let $F$ be an extended real-membered set and let $f$ be an extended real number. One can check that $F \odot f$ is extended real-membered.

The following propositions are true:

(229) If $a \neq 0$ and $A \odot a \subseteq B \odot a$, then $A \subseteq B$.
(230) If $a \neq 0$ and $A \odot a = B \odot a$, then $A = B$.
(231) If $a \neq 0$, then $A \cap B \odot a = (A \odot a) \cap (B \odot a)$.
(232) If $a \neq 0$, then $(A \setminus B) \odot a = (A \odot a) \setminus (B \odot a)$.
(233) If $a \neq 0$, then $(A \setminus B) \odot a = (A \odot a) \setminus (B \odot a)$.
(234) $(A \oplus B) \odot a = (A \odot a) \oplus (B \odot a)$.
(235) $(A \ominus B) \odot a = (A \odot a) \ominus (B \odot a)$.
REFERENCES


Received December 19, 2008