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Darboux's Theorem

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Summary. In this article, we have proved the Darboux's theorem. This theorem is important to prove the Riemann integrability. We can replace an upper bound and a lower bound of a function which is the definition of Riemann integration with convergence of sequence by Darboux's theorem.

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The articles [20], [22], [2], [21], [11], [23], [4], [5], [24], [12], [6], [8], [3], [15], [7], [9], [14], [13], [17], [19], [18], [16], [1], and [10] provide the notation and terminology for this paper.

1. Lemmas of Division

We adopt the following rules: x, y are real numbers, i, j, k are natural numbers, and p, q are finite sequences of elements of $\mathbb{R}$.

We now state a number of propositions:

(1) Let A be a closed-interval subset of $\mathbb{R}$ and D be an element of $\operatorname{divs} A$. If $\operatorname{vol}(A) \neq 0$, then there exists i such that $i \in \operatorname{dom} D$ and $\operatorname{vol}(\operatorname{divset}(D,i)) > 0$.

(2) Let A be a closed-interval subset of $\mathbb{R}$ and D be an element of $\operatorname{divs} A$. If $x \in A$, then there exists j such that $j \in \operatorname{dom} D$ and $x \in \operatorname{divset}(D, j)$.

(3) Let A be a closed-interval subset of $\mathbb{R}$ and D_1, D_2 be elements of $\operatorname{divs} A$. Then there exists an element D of $\operatorname{divs} A$ such that $D_1 \leq D$ and $D_2 \leq D$ and $\operatorname{rng} D = \operatorname{rng} D_1 \cup \operatorname{rng} D_2$.

(4) Let A be a closed-interval subset of $\mathbb{R}$ and D, D_1 be elements of $\operatorname{divs} A$. Suppose $\delta_{(D_1)} < \min \operatorname{rng} \operatorname{upper\_volume}(\chi_{A,A}, D)$. Let given x, y, i. If $i \in \operatorname{dom} D_1$ and $x \in \operatorname{rng} D \cap \operatorname{divset}(D_1, i)$ and $y \in \operatorname{rng} D \cap \operatorname{divset}(D_1, i)$, then $x = y$.

(5) For all p, q such that $\operatorname{rng} p = \operatorname{rng} q$ and p is increasing and q is increasing holds $p = q$.

(6) Let A be a closed-interval subset of $\mathbb{R}$ and D, D_1 be elements of $\operatorname{divs} A$. If $D \leq D_1$ and $i \in \operatorname{dom} D$ and $j \in \operatorname{dom} D$ and $i \leq j$, then $\operatorname{indx}(D_1, D, i) \leq \operatorname{indx}(D_1, D, j)$ and $\operatorname{indx}(D_1, D, i) \in \operatorname{dom} D_1$.

(7) Let A be a closed-interval subset of $\mathbb{R}$ and D, D_1 be elements of $\operatorname{divs} A$. If $D \leq D_1$ and $i \in \operatorname{dom} D$ and $j \in \operatorname{dom} D$ and $i < j$, then $\operatorname{indx}(D_1, D, i) < \operatorname{indx}(D_1, D, j)$.

(8) For every closed-interval subset A of $\mathbb{R}$ and for every element D of $\operatorname{divs} A$ holds $\delta_D \geq 0$.

(9) Let A be a closed-interval subset of $\mathbb{R}$, g be a function from A into $\mathbb{R}$, and D_1, D_2 be elements of divs$A$. Suppose $x \in \operatorname{divset}(D_1, \operatorname{len} D_1)$ and $\operatorname{len} D_1 \geq 2$ and $D_1 \leq D_2$ and $\operatorname{rng} D_2 = \operatorname{rng} D_1 \cup \{x\}$ and g is bounded on A. Then $\sum \text{lower\_volume}(g, D_2) - \sum \text{lower\_volume}(g, D_1) \leq (\sup \operatorname{rng} g - \inf \operatorname{rng} g) \cdot \delta_{(D_1)}$.

(10) Let A be a closed-interval subset of $\mathbb{R}$, g be a function from A into $\mathbb{R}$, and D_1, D_2 be elements of divs$A$. Suppose $x \in \operatorname{divset}(D_1, \operatorname{len} D_1)$ and $\operatorname{len} D_1 \geq 2$ and $D_1 \leq D_2$ and $\operatorname{rng} D_2 = \operatorname{rng} D_1 \cup \{x\}$ and g is bounded on A. Then $\sum \text{upper\_volume}(g, D_1) - \sum \text{upper\_volume}(g, D_2) \leq (\sup \operatorname{rng} g - \inf \operatorname{rng} g) \cdot \delta_{(D_1)}$.

(11) Let A be a closed-interval subset of $\mathbb{R}$, D be an element of divs$A$, r be a real number, and i, j be natural numbers. Suppose $i \in \operatorname{dom} D$ and $j \in \operatorname{dom} D$ and $i \leq j$ and $r < (\operatorname{mid}(D, i, j))(1)$. Then there exists a closed-interval subset B of $\mathbb{R}$ such that $r = \inf B$ and $\sup B = (\operatorname{mid}(D, i, j))(\operatorname{len} \operatorname{mid}(D, i, j))$ and $\operatorname{len} \operatorname{mid}(D, i, j) = (j - i) + 1$ and $\operatorname{mid}(D, i, j)$ is a DivisionPoint of B.

(12) Let A be a closed-interval subset of $\mathbb{R}$, f be a function from A into $\mathbb{R}$, and D_1, D_2 be elements of divs$A$. Suppose $x \in \operatorname{divset}(D_1, \operatorname{len} D_1)$ and $\operatorname{vol}(A) \neq 0$ and $D_1 \leq D_2$ and $\operatorname{rng} D_2 = \operatorname{rng} D_1 \cup \{x\}$ and f is bounded on A and $x > \inf A$. Then $\sum \text{lower\_volume}(f, D_2) - \sum \text{lower\_volume}(f, D_1) \leq (\sup \operatorname{rng} f - \inf \operatorname{rng} f) \cdot \delta_{(D_1)}$.

(13) Let A be a closed-interval subset of $\mathbb{R}$, f be a function from A into $\mathbb{R}$, and D_1, D_2 be elements of divs$A$. Suppose $x \in \operatorname{divset}(D_1, \operatorname{len} D_1)$ and $\operatorname{vol}(A) \neq 0$ and $D_1 \leq D_2$ and $\operatorname{rng} D_2 = \operatorname{rng} D_1 \cup \{x\}$ and f is bounded on A and $x > \inf A$. Then $\sum \text{upper\_volume}(f, D_1) - \sum \text{upper\_volume}(f, D_2) \leq (\sup \operatorname{rng} f - \inf \operatorname{rng} f) \cdot \delta_{(D_1)}$.

(14) Let A be a closed-interval subset of $\mathbb{R}$, D_1, D_2 be elements of divs$A$, r be a real number, and i, j be natural numbers. Suppose $i \in \operatorname{dom} D_1$ and $j \in \operatorname{dom} D_1$ and $i \leq j$ and $D_1 \leq D_2$ and $r < (\operatorname{mid}(D_2, \operatorname{indx}(D_2, D_1, i), \operatorname{indx}(D_2, D_1, j)))(1)$. Then there exists a closed-interval subset B of $\mathbb{R}$ and there exist elements M_1, M_2 of divs$B$ such that $r = \inf B$ and $\sup B = M_2(\operatorname{len} M_2)$ and $\sup B = M_1(\operatorname{len} M_1)$ and $M_1 \leq M_2$ and $M_1 = \operatorname{mid}(D_1, i, j)$ and $M_2 = \operatorname{mid}(D_2, \operatorname{indx}(D_2, D_1, i), \operatorname{indx}(D_2, D_1, j))$.

(15) For every closed-interval subset A of $\mathbb{R}$ and for every element D of divs$A$ such that $x \in \operatorname{rng} D$ holds $D(1) \leq x$ and $x \leq D(\operatorname{len} D)$.

(16) Let p be a finite sequence of elements of $\mathbb{R}$ and given i, j, k. Suppose p is increasing and $i \in \operatorname{dom} p$ and $j \in \operatorname{dom} p$ and $k \in \operatorname{dom} p$ and $p(i) \leq p(k)$ and $p(k) \leq p(j)$. Then $p(k) \in \operatorname{rng} \operatorname{mid}(p, i, j)$.

(17) Let A be a closed-interval subset of $\mathbb{R}$, f be a function from A into $\mathbb{R}$, and D be an element of divs$A$. If f is bounded on A and $i \in \operatorname{dom} D$, then $\inf \operatorname{rng}(f \restriction \operatorname{divset}(D, i)) \leq \sup \operatorname{rng} f$.

(18) Let A be a closed-interval subset of $\mathbb{R}$, f be a function from A into $\mathbb{R}$, and D be an element of divs$A$. If f is bounded on A and $i \in \operatorname{dom} D$, then $\sup \operatorname{rng}(f \restriction \operatorname{divset}(D, i)) \geq \inf \operatorname{rng} f$.

2. Darboux's Theorem

Next we state two propositions:

(19) Let A be a closed-interval subset of $\mathbb{R}$, f be a function from A into $\mathbb{R}$, and T be a DivSequence of A. Suppose f is bounded on A and δ_T is convergent to 0 and $\operatorname{vol}(A) \neq 0$. Then $\text{lower\_sum}(f, T)$ is convergent and $\lim \text{lower\_sum}(f, T) = \text{lower\_integral} f$.

(20) Let A be a closed-interval subset of $\mathbb{R}$, f be a function from A into $\mathbb{R}$, and T be a DivSequence of A. Suppose f is bounded on A and δ_T is convergent to 0 and $\operatorname{vol}(A) \neq 0$. Then $\text{upper\_sum}(f, T)$ is convergent and $\lim \text{upper\_sum}(f, T) = \text{upper\_integral} f$.

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